

This discussion paper is/has been under review for the journal Hydrology and Earth System Sciences (HESS). Please refer to the corresponding final paper in HESS if available.

Which type of slope gradient should be used to determine flow-partition proportion in multiple-flow-direction algorithms – tangent or sine?

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Received: 8 April 2012 – Accepted: 14 May 2012 – Published: 24 May 2012

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Published by Copernicus Publications on behalf of the European Geosciences Union.

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Abstract

This paper revisits the tangent expression for the slope gradient ($\tan\beta$) used in the general flow-partition function in multiple-flow-direction (MFD) algorithms. The deduction of the flow-partition function performed here based on hydrological theory shows that $\sin\beta$, instead of $\tan\beta$ as used currently, provides a clear theoretical basis for determining the flow-partition proportions in MFD algorithms.

1 Introduction

The flow-direction algorithm is one of the most important types of algorithms in digital terrain analysis and is widely used in many terrain-related applications such as hydrologic analysis, soil erosion, and geomorphology (Wilson, 2012; Hengl and Reuter, 2008). Most existing flow-direction algorithms use a gridded digital elevation model (DEM) as their basic input to determine the flow partitioning of each cell in the DEM among its neighboring cell(s) (Wilson, 2012).

Based on whether it is assumed that the flow along the slope surface can be drained from a cell into only one neighboring cell or into more than one downslope neighboring cell, existing flow-direction algorithms are classified into two main types: single-flow-direction (SFD) algorithms (e.g. O'Callaghan and Mark (1984)'s D8 algorithm) and multiple-flow-direction (MFD) algorithms (e.g. Quinn et al., (1991)'s FD8 algorithm and Qin et al. (2007)'s MFD-md algorithm). Generally, MFD performs better than SFD, especially when the flow-direction algorithm is used to derive the spatial pattern of hydrological parameters (such as specific catchment area and topographic wetness index) at a fine scale (Wolock and McCabe, 1995; Qin et al., 2011; Wilson, 2012).

The key issue in MFD is how to partition the flow into multiple downslope cells. The general principle for determining the flow-partition proportions is based on the hydraulic gradient, according to which a neighboring cell with a steeper descent should drain more flow from the central cell (Quinn et al., 1991). The hydraulic gradient is normally

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approximated using a type of slope gradient (i.e. the tangent value of the slope angle, $\tan\beta$) in the practical flow-partition functions used in most MFD algorithms (Qin et al., 2007), as is the case in the following flow-partition function used in the classic FD8 algorithm proposed by Quinn et al. (1991):

$$d_i = \frac{(\tan\beta_i)^p \times L_i}{\sum_{j=1}^8 [(\tan\beta_j)^p \times L_j]}, \quad (1)$$

where d_i is the fraction of flow draining into the i -th neighboring cell from a given cell, $\tan\beta_i$ is the slope gradient between the central cell and the i -th neighboring cell, p is the flow-partition exponent, and L_i is the “effective contour length” for the i -th neighboring cell, which is $1/2$ for downslope cells in cardinal directions, $\sqrt{2}/4$ for downslope cells in diagonal directions, and 0 for non-downslope neighboring cells. Efforts to revise Eq. (1) to obtain a more reasonable result, as made in many MFD algorithms derived from FD8, have been focused on how to set the parameters p (e.g. Holmgren, 1994; Quinn et al., 1995, 2007) and L_i (e.g. Chirico et al., 2005).

However, there has been little discussion of the theoretical basis and the scope of application for approximating the hydraulic gradient using $\tan\beta$ in both the flow-partition function as Eq. (1) and in other derived flow-partition functions for MFD algorithms, although $\tan\beta$ obviously plays a key role in the flow-partition function for MFD algorithms. In this paper, the general flow-partition function is deduced based on hydrological theory, and approximation of the hydraulic gradient using $\tan\beta$ to determine the flow-partition proportions in existing MFD algorithms is found to be questionable.

2 Deduction of the flow-partition function

This section describes the deduction of the flow-partition proportion based on hydrological theory, including a review of both the theoretical basis and the physical meaning

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of the flow-partition function. The outflow from a central cell to its i -th downslope neighboring cell, Q_i , can be expressed as:

$$Q_i = A_i \times V_i, \quad (2)$$

where A_i is the cross-sectional area of the partition between the central cell and its i -th downslope neighboring cell and V_i is the flow rate in the direction of the i -th downslope neighboring cell. Q_i is zero for non-downslope neighboring cells.

The flow-partition proportion can be deduced from Eq. (2) under two cases of outflow: surface flow and subsurface flow. If the outflow is surface flow, the surface flow rate can be described by the Manning equation (Chow et al., 1988):

$$V = \frac{1}{n} R^{2/3} S^{1/2}, \quad R = \frac{A}{L}, \quad (3)$$

where n is the Manning roughness of the surface ($\text{s m}^{-1/3}$), R is the hydraulic radius (m), L is the wetted perimeter (m), and S is the hydraulic gradient. Substituting Eq. (3) into Eq. (2) yields:

$$Q_i = A_i \times V_i = \frac{1}{n_i} L_i R_i^{5/3} S_i^{1/2}, \quad (4)$$

By further assuming that the Manning roughness n_i and the hydraulic radius R_i are constant in all downslope directions of a given cell, the flow-partition proportion can be expressed as follows:

$$d_i = \frac{Q_i}{\sum_{j=1}^8 Q_j} = \frac{\frac{1}{n_i} L_i R_i^{5/3} S_i^{1/2}}{\sum_{j=1}^8 \frac{1}{n_j} L_j R_j^{5/3} S_j^{1/2}} = \frac{S_i^{1/2} \times L_i}{\sum_{j=1}^8 S_j^{1/2} \times L_j} \quad \text{for all } S_j > 0, \quad (5)$$

where L_i is the wetted perimeter, which can be interpreted as “effective contour length” in a gridded DEM, and S_i is the hydraulic gradient in the direction of the i -th neighboring cell. This is the case of Eq. (1) when the flow-partition exponent p is 1/2.

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If the outflow is subsurface flow, Darcy's equation can be used to describe the subsurface flow rate, which is linearly proportional to the hydraulic gradient (Chow et al., 1988), i.e.:

$$V = k \times S, \quad (6)$$

5 where k is the hydraulic conductivity and S is the hydraulic gradient. Thus, the flow-partition proportion can be expressed as follows:

$$d_j = \frac{S_j \times L_j}{\sum_{j=1}^8 (S_j \times L_j)} \text{ for all } S_j > 0 \quad (7)$$

This is the case of Eq. (1) with the flow-partition exponent p set to one.

10 According to Eqs. (5) and (7), the flow-partition function is based on the hydraulic gradient (S), which can be formalized as a function of flow direction (λ) and slope angle (β) (Fig. 1a) (Montgomery and Dietrich, 1994; Ghiassian and Ghareh, 2008):

$$S = \frac{\sin(\beta)}{\sin(\lambda)} \quad (8)$$

15 In MFD algorithms using the flow-partition function, the flow is assumed to run along the slope, that is, the flow direction is parallel to the slope surface ($\lambda = \pi/2$, as shown in Fig. 1b). Thus, the hydraulic gradient can be expressed as:

$$S = \frac{\sin(\beta)}{\sin(\lambda)} = \frac{\sin(\beta)}{\sin(\pi/2)} = \sin(\beta). \quad (9)$$

The above deduction shows that the flow-partition function should be conceptually related to the hydraulic gradient, which is derived as a function of $\sin\beta$ during the application of the flow-partition function.

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3 Discussion

It is unreasonable that the tangent of the slope angle ($\tan\beta$), instead of the sine of the slope angle ($\sin\beta$), is used to determine the flow-partition proportions (Eq. 1) in most current MFD algorithms because $\sin\beta$ provides a clear theoretical basis for the flow-partition function, as shown in the above deduction.

Why does this unreasonable situation occur with the flow-partition function? The authors believe that it is mainly because the design of the flow-partition function for primary MFD algorithms continues to use the slope gradient as in earlier SFD algorithms. SFD algorithms aim to find the neighboring cell with the steepest downslope from each cell and then to drain all flow from the central cell into this neighboring cell. Therefore, the use of $\tan\beta$ or $\sin\beta$ makes no difference in SFD algorithms. Compared with $\sin\beta$, $\tan\beta$ is more conventionally and widely used to define the slope gradient. Therefore, $\tan\beta$ is traditionally used to determine the flow direction in SFD algorithms. It is intuitive and natural to use $\tan\beta$ to express the slope gradient in the flow-partition function for MFD algorithms, as in SFD algorithms. However, the deduction performed in the previous section shows that $\sin\beta$, instead of $\tan\beta$, should be used to determine the flow-partition proportions for MFD algorithms.

How will the use of $\sin\beta$ instead of $\tan\beta$ in the flow-partition function affect the results of MFD algorithms? We applied both the original version (using $\tan\beta$ in the flow-partition function) of three typical MFD algorithms and their sine-version (using $\sin\beta$ instead of $\tan\beta$) to two areas of 3×3 cells at 10-m resolution (Fig. 2). The three MFD algorithms include Quinn et al. (1991)'s FD8 algorithm, Holmgren (1994)'s FMFD algorithm with the flow-partition exponent (ρ) being 5, and Qin et al. (2007)'s MFD-md algorithm. The two areas are with lower relief (the maximum downslope angle is less than 10° , Fig. 2a) and higher relief (the maximum downslope angle is about 35° , Fig. 2b), respectively. When the difference between $\tan\beta$ and $\sin\beta$ is subtle in low-relief terrain conditions, the difference between results from the original version and the sine-version of each tested MFD algorithm is also subtle (Fig. 2a). When the difference between

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$\tan\beta$ and $\sin\beta$ increases dramatically as the terrain becomes steeper, a flow-partition proportion derived from the function based on $\tan\beta$ deviates much more markedly from the result obtained from the function based on $\sin\beta$ (Fig. 2b). In the high-relief case, maximum deviations are close to 10% for MFD-md and FMFD algorithms, and less than 3% for FD8 algorithm, respectively. The higher the flow-partition exponent ρ , the greater is the deviation when $\sin\beta$ instead of $\tan\beta$ is used in an MFD algorithm. This phenomenon can be attributed to the form of Eq. (1). A detailed discussion on the effect of using $\sin\beta$ instead of $\tan\beta$ in MFD algorithms still needs additional work by using real or synthetic landscapes.

4 Conclusions

This paper revisits the use of $\tan\beta$ in the general flow-partition function for MFD algorithms. A deduction of the flow-partition function based on hydrological theory shows that $\sin\beta$, instead of $\tan\beta$ as used currently, provides a clear theoretical basis for determining the flow-partition proportions in MFD algorithms. Both a new flow-partition function based on $\sin\beta$ and a corresponding new MFD algorithm should be designed. When applying existing MFD algorithms, users should be especially prudent in interpreting the results in study areas with steep slopes.

Acknowledgements. This study was partially funded by the Chinese Academy of Sciences under Grant No. KZCX2-YW-Q10-1-5, and by the National Natural Science Foundation of China (No. 40971235). Supports from the Institute of Geographic Sciences and Natural Resources Research, CAS are also appreciated. We thank Rulin Ouyang and Yichi Zhang for their comments on the deduction in this study. We also thank Ross Woods for his kindly handling of the review process of this paper and his constructive comments.

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		(a) low-relief case			(b) high-relief case		
Elevation (at 10-m resolution)		101	101	101	101	101	101
		101	100	99	101	100	96
		101	99.5	98	101	95	90
Downslope				5.7°			21.8°
			2.9°	8°		26.6°	35.3°
<i>FD8</i>	Original version		40%			28.57%	
			20%	40%		35.71%	35.72%
	Sine-version		40.05%			30.27%	
			20.10%	39.85%		36.45%	33.28%
<i>FMFD (p=5)</i>	Original version		14.95%			4.69%	
			0.47%	84.58%		14.32%	80.99%
	Sine-version		15.26%			7.93%	
			0.49%	84.25%		20.08%	71.99%
<i>MFD-md</i>	Original version		35.76%			1.86%	
			6.97%	57.27%		9.65%	88.49%
	Sine-version		35.97%			4.27%	
			7.08%	56.95%		16.88%	78.85%

Fig. 2. Examples of the calculated flow-partition proportions when $\sin \beta$ (sine-version) and $\tan \beta$ (original version) are used in typical MFD algorithms, respectively, under **(a)** the low-relief case, and **(b)** the high-relief case.

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